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LETTER TO THE EDITOR

Non-linear wave equations in a curved background space

P M Radmore and G Stephenson

Department of Mathematics, Imperial College, London, UK

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Abstract. Derrick's theorem concerning the existence of soliton-like solutions of non-linear scalar wave equations in Minkowski space is extended to the curved background space exterior to a charged, non-rotating black hole.

In a recent series of papers (Rowan and Stephenson, 1976a, b, 1977, Rowan 1977, Radmore 1978) solutions of the Klein–Gordon scalar wave equation in curved background spaces were obtained using Liouville–Green techniques. These solutions were related to the infall of baryons into black holes. We now consider whether it is possible to have soliton-like solutions of the *non-linear* Klein–Gordon equation containing self-interaction terms in the space exterior to a charged, non-rotating black hole as described by the Reissner–Nordström metric. It is well-known (Derrick 1964) that if Φ is a scalar field in one time and D space dimensions satisfying the non-linear equation

$$\frac{\partial^2\Phi}{\partial t^2}-\nabla^2\Phi=-\tfrac{1}{2}f'(\Phi) \tag{1}$$

derivable from the variational principle

$$\delta\int[(\partial\Phi/\partial t)^2-(\nabla\Phi)^2-f(\Phi)]\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}t=0 \tag{2}$$

then for $D\geqslant 2$ and $f(\Phi)\geqslant 0$ the only non-singular time-independent solutions are the vacuum (or ground) states for which $f(\Phi)=0$. This result, however, was established only in Minkowski space and we now extend this work to the space exterior to a non-rotating black hole of mass m and charge e defined by the metric

$$\mathrm{d}s^2=\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)\mathrm{d}t^2-\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)^{-1}\mathrm{d}r^2-r^2\,\mathrm{d}\theta^2-r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{3}$$

We first write (1) in covariant form as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,g^{ik}\frac{\partial\Phi}{\partial x^k}\right)=-\tfrac{1}{2}f'(\Phi) \tag{4}$$

which arises from the variational principle

$$\delta\int\left(g^{ik}\frac{\partial\Phi}{\partial x^i}\frac{\partial\Phi}{\partial x^k}-f(\Phi)\right)\sqrt{-g}\,\mathrm{d}^4x=0. \tag{5}$$

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Using (3) and taking Φ to be a function of r only, we obtain from (4) the radial equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left((r^2-2mr+e^2)\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)=\tfrac{1}{2}f'(\Phi). \tag{6}$$

In this case the variational principle (5) is equivalent to

$$\delta E=0, \tag{7}$$

where the energy E of the Φ field is given by

$$E=4\pi\int_{r_+}^{\infty}\left[(r^2-2mr+e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2+f(\Phi)r^2\right]\mathrm{d}r, \tag{8}$$

and where $r_+=m+\sqrt{(m^2-e^2)}$ $(e^2\leqslant m^2)$ is the event horizon of the black hole.

Writing

$$I_1=\int_{r_+}^{\infty}(r^2-2mr+e^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r \tag{9}$$

and

$$I_2=\int_{r_+}^{\infty}f(\Phi)r^2\,\mathrm{d}r \tag{10}$$

so that

$$E=4\pi(I_1+I_2) \tag{11}$$

we must require that I_1 and I_2 converge.

We now define

$$\Phi_\alpha(r)=\Phi(\alpha r), \tag{12}$$

where α is an arbitrary constant and

$$E_\alpha=4\pi\int_{r_+}^{\infty}\left((r^2-2mr+e^2)\left(\frac{\mathrm{d}\Phi_\alpha}{\mathrm{d}r}\right)^2+f(\Phi_\alpha)r^2\right)\mathrm{d}r. \tag{13}$$

Then on changing the variable of integration from r to αr we have

$$\frac{E_\alpha}{4\pi}=\int_{\alpha r_+}^{\infty}(r^2-2m\alpha r+e^2\alpha)\frac{1}{\alpha}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r+\int_{\alpha r_+}^{\infty}f(\Phi)\frac{r^2}{\alpha^3}\,\mathrm{d}r. \tag{14}$$

Differentiation of (14) with respect to α gives

$$\frac{1}{4\pi}\frac{\mathrm{d}E_\alpha}{\mathrm{d}\alpha}=\int_{\alpha r_+}^{\infty}(-2mr+2e^2\alpha^2)\frac{1}{\alpha}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r+\int_{\alpha r_+}^{\infty}(r^2-2m\alpha r+e^2\alpha^2)\left(-\frac{1}{\alpha^2}\right)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2\mathrm{d}r$$
$$+\int_{\alpha r_+}^{\infty}f(\Phi)\left(-\frac{3r^2}{\alpha^4}\right)\mathrm{d}r-\frac{r_+^3}{\alpha}f(\Phi)\bigg|_{r=\alpha r_+} \tag{15}$$

so that

$$\frac{1}{4\pi}\frac{\mathrm{d}^2E_\alpha}{\mathrm{d}\alpha^2}\bigg|_{\alpha=1}=-I_1-3I_2+I_3-r_+^3f(\Phi)\bigg|_{r=r_+} \tag{16}$$

where

$$I_3 = \int_{r_+}^{\infty} (-2mr + 2e^2)\left(\frac{d\Phi}{dr}\right)^2 dr. \tag{17}$$

Now from (7) we must have

$$\left.\frac{dE_\alpha}{d\alpha}\right|_{\alpha=1} = 0 \tag{18}$$

which gives from (16)

$$3I_2 = -I_1 + I_3 - r_+^3 f(\Phi)\Big|_{r=r_+} \tag{19}$$

Similarly, differentiating (14) twice with respect to α and setting $\alpha = 1$ we obtain

$$\frac{1}{4\pi}\left.\frac{d^2E_\alpha}{d\alpha^2}\right|_{\alpha=1} = I_4 - 2I_3 + 2I_1 + 12I_2 + 4r_+^3 f(\Phi)\Big|_{r=r_+}$$
$$+ r_+(2mr_+ - 2e^2)\left(\frac{d\Phi}{dr}\right)^2\Big|_{r=r_+} - r_+^4\left[f'(\Phi)\frac{d\Phi}{dr}\right]_{r=r_+} \tag{20}$$

where

$$I_4 = \int_{r_+}^{\infty} 2e^2\left(\frac{d\Phi}{dr}\right)^2 dr. \tag{21}$$

Now using (19) we eliminate I_2 from (20) to get

$$\frac{1}{4\pi}\left.\frac{d^2E_\alpha}{d\alpha^2}\right|_{\alpha=1} = I_4 + 2I_3 - 2I_1 + r_+(2mr_+ - 2e^2)\left(\frac{d\Phi}{dr}\right)^2\Big|_{r=r_+} - r_+^4\left[f'(\Phi)\frac{d\Phi}{dr}\right]_{r=r_+}. \tag{22}$$

From (6) we have

$$\left[f'(\Phi)\frac{d\Phi}{dr}\right]_{r=r_+} = \frac{4(r_+ - m)}{r_+^2}\left(\frac{d\Phi}{dr}\right)^2\Big|_{r=r_+} \tag{23}$$

and substitution of (23) into (22) leads to

$$\frac{1}{4\pi}\left.\frac{d^2E_\alpha}{d\alpha^2}\right|_{\alpha=1} = I_4 + 2I_3 - 2I_1 - 2r_+(mr_+ - e^2)\left(\frac{d\Phi}{dr}\right)^2\Big|_{r=r_+}. \tag{24}$$

Finally, inserting the expressions for I_1, I_3 and I_4 from (9), (17) and (21), equation (24) becomes

$$\frac{1}{8\pi}\left.\frac{d^2E_\alpha}{d\alpha^2}\right|_{\alpha=1} = \int_{r_+}^{\infty} (2e^2 - r^2)\left(\frac{d\Phi}{dr}\right)^2 dr - r_+(mr_+ - e^2)\left(\frac{d\Phi}{dr}\right)^2\Big|_{r=r_+}. \tag{25}$$

A necessary condition for the solution of (6) to be stable is

$$\left.\frac{d^2E_\alpha}{d\alpha^2}\right|_{\alpha=1} \geqslant 0 \tag{26}$$

which from (25) is

$$\int_{r_+}^{\infty} (2e^2 - r^2)\left(\frac{d\Phi}{dr}\right)^2 dr - r_+(mr_+ - e^2)\left(\frac{d\Phi}{dr}\right)^2\Big|_{r=r_+} \geqslant 0. \tag{27}$$

We can now establish a general result. If $f(\Phi)\geqslant 0$ then from (10)

$$I_2\geqslant 0. \tag{28}$$

We also have from (9) and (17)

$$I_1\geqslant 0, \qquad I_3\leqslant 0. \tag{29}$$

On substituting (28) and (29) into (19) we see that we must have (since $f(\Phi)\geqslant 0$)

$$I_1=I_3=f(\Phi)=0 \tag{30}$$

giving that the only solutions of (6) are those where Φ is a constant C satisfying $f(C)=0$.

We now consider two special cases. Firstly, suppose that

$$\tfrac{1}{2}f'(\Phi)=\lambda\Phi^3+\mu^2\Phi, \qquad \lambda, \mu \text{ constant.} \tag{31}$$

Since then $f(\Phi)=\frac{1}{2}\lambda\Phi^4+\mu^2\Phi^2$ is non-negative, (30) gives that (6) has only the trivial solution $\Phi=0$. Secondly, suppose that

$$\tfrac{1}{2}f'(\Phi)=\lambda\Phi^3-\mu^2\Phi \tag{32}$$

which is the form of current interest in gauge theories. Then again

$$f(\Phi)=\tfrac{1}{2}\lambda[\Phi^2-(\mu^2/\lambda)]^2 \tag{33}$$

is non-negative. The only solutions of (6) are therefore the vacuum states

$$\Phi=\pm\mu/\sqrt{\lambda}. \tag{34}$$

For compact spatial topologies there may well exist non-trivial stable vacuum solutions (Avis and Isham 1978).

Finally, if no restriction is made on the sign of $f(\Phi)$, then we may have non-constant, finite energy solutions of (6). If (27) is to hold for such solutions, we must have $2e^2-r^2>0$ for some part of the range $r_+\leqslant r<\infty$ since the second term in (27) is non-positive. This gives $\sqrt{2}e>r_+$ or

$$m^2>e^2>\tfrac{8}{9}m^2. \tag{35}$$

In particular (35) shows that there will be no such solutions in a Schwarzschild background space.

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